REVIEWS

Differential—und integral—ungleichungen. By W. Walter. Springer, 1964. 269 pp. DM. 59.

This book consists of a set of highly fruitful variations on a single theme. The theme is a familiar one in analysis: the estimation *a priori* of solutions of a nonlinear differential or integral equation in terms of solutions of a corresponding linear equation or linear inequality. Nowhere else, however, has this idea been carried out systematically, one might almost say relentlessly, for such a broad class of practically interesting problems.

At the core of the work is a series of estimation theorems ('Abschätzungssätze') of remarkably consistent form. In each case the solution u of a nonlinear problem $Tu = 0$ is sought; an approximate solution v is assumed known, and a class of error estimates $|u-v| \leqslant \rho$ is derived. The functions ρ are determined only implicitly, as solutions of a 'substitute problem' $\Omega\rho = 0$ or sometimes $\Omega\rho > 0$. The (possibly inhomogeneous) linear operator Ω is explicitly expressed in terms of two quantities: an upper bound δ on the residual $|Tv|$, and some sort of modulus of continuity for the 'nonlinear part' of T.

The simplest such modulus of continuity would be a Lipschitz condition, but more general moduli are frequently used. Indeed, for maximum utility it is desirable to restrict Ω as little as possible, since one then obtains the largest class of error functions ρ and therefore the closest estimates for u. To accomplish this, the author often hypothesizes only one-sided moduli of continuity or moduli applicable only to solutions.

With just one exception, uniqueness theorems are derived throughout from error estimates, rather than directly. Thus they automatically carry with them theorems on continuous dependence. Existence theorems, on the other hand, are proved only for Volterra integral equations.

Volterra equations in one independent variable, with their usual application to initial value problems for ordinary differential equations, form the subject of chapter I. The moduli of continuity here come from integral estimates, and the proofs of the Abschätzungssätze are based on Gronwall's Lemma. Particularly elegant results are obtained for Volterra operators with monotone kernels.

Chapter II is entitled Ordinary Differential Equations and deals with estimation methods for initial value problems based on geometrical properties of the solution-field. The fundamental idea is simply that neighbouring solutions cannot coalesce. This idea is extended in a neat way to systems of differential equations. The notation of a K-norm is introduced to facilitate systematic treatment of one-sided moduli of continuity, which often yield exponentially decreasing error estimates. Applications worth special mention are a two-stage uniqueness proof due to Krasnoselski and Krein (14IV) and a surprisingly close estimate for solutions of the Blasius equation (15XIII).

The third chapter treats Volterra equations in several variables. The results are applied to hyperbolic partial differential equations of the form $u_{xy} = f(x,y,u)$ and, more generally, $u_{xy} = f(x,y,u,u_x,u_y)$. The analogy with ordinary differential equations is pushed as far as possible, u_{xy} merely taking the place of u', but often extra monotonicity conditions must be imposed on f. Disappointingly, estimates from one-sided moduli are not included.

A highlight of this chapter is the construction (21 IV), starting from characteristic ordinary differential equations with multiple solutions, of a hyperbolic initial-value problem without any solution at all. Also of note is a new existence theorem (18 V) for systems of Volterra equations in several dimensions, the proof of which is connected with the problem of uniqueness in fewer dimensions. Incidentally, this theorem seems to have an unnecessarily strong hypothesis: continuity in the integration-variable ξ is needed only in directions perpendicular to the hyperplane of integration.

The fourth and last chapter, entitled Parabolic Equations, contains major applications of interest to the fluid dynamicist. The basic form of differential equation treated is $u_t = f(t,x,u,u_x,u_{xx})$, where $\partial f/\partial u_{xx} > 0$, with an initial condition and (perhaps) boundary conditions. The methods, based mostly on the local maximum principle, are direct extensions of those used in Chapter II. The strong maximum principle for uniformly parabolic equations (26 IV), the Rayleigh–Ritz estimation procedure (26 VIII), stability results of Phragmen–Lindelof type (28 XII) all come to light as mere special cases of the systematically derived Abschätzungssätze. Another special case is a theorem of Nickel (26 VI) with an elegant geometrical interpretation: the solution remains within the convex hull of the graph of the initial and boundary data. Systems of parabolic equations are also treated, and some new one-sided estimation theorems (32 VII, X) are derived.

Extensive sections in chapter IV are devoted to two specific applications: the heat equation (with variable coefficients) and the equations of boundary-layer theory. For the heat equation, an unusual example is given in which a physically obvious minimum-principle is mathematically false unless the additional condition of non-negative temperature is imposed.

The treatment of stationary boundary-layer theory, though confined to the two-dimensional case, leaves little to be desired in elegance and economy. Uniqueness is established by a one-sided Lipschitz condition in the absence of backward flow, then backward flow is beautifully excluded through the use of an Abschätzungssatz one of whose hypotheses, an estimate from below by the Blasius solution, is not even valid in the situations one wishes to exclude!

The time-dependent boundary-layer problem is also discussed, as an example of a 'generalized' parabolic equation with several 'time-like' variables (in this case two, the downstream coordinate and the physical time). To fit the problem into this mould, however, the continuity equation is dropped, so that the estimates obtained are probably over-generous, while uniqueness cannot be proved at all.

Conspicuously missing from the book, though intimately related to its subject, are Green's functions, energy inequalities, and all but the most perfunctory discussion of elliptic equations. The author sticks to his self-imposed framework, treating only problems involving some sort of initial condition. Thus the opportunity to present a systematic derivation of the weak and strong maximum principles for elliptic equations is not grasped.

Another unfortunate feature is the minimal role played by numerical methods. While the author does show (25 XIII) how partial difference equations can be reduced to systems of ordinary differential equations and so brought within the scope of chapter II, he neglects almost completely the systematic employment of difference equations to create approximate solutions v and error functions ρ in his estimation theorems. The one very tantalizing exception is the Blasius equation (15 XIII) mentioned above.

Professor Walter writes with clarity and precision. He takes good care to introduce the principal ideas gradually, with many mutually reinforcing examples, and to develop them in harmony and balance. He succeeds in welding the known results on uniqueness and continuous dependence into a single structural unit, with a consistent notation throughout. Regrettably, however, an over-condensed, forbidding summary of this notation is placed at the very beginning of the book, where it tends to frighten off the casual reader.

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